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Cohen-Macaulay Graphs

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Abstract

In this article the author present two methods meant to construct Cohen-Macaulay graphs and some interesting examples and properties. This paper also presents an important property of Cohen-Macalay ring to have Cohen-Macaulay fiber.

Key words: *Cohen-Macaulay graph, Cohen-Macaulay ring, edge ideal*

Notation and Definitions

Let G be a graph on the vertex set $V = \{v_1, ..., v_n\}$, $E(G)$ the edge set of G, $R = k[x_1, ..., x_n]$ the polynomial ring over a field k, we will often identify the vertex v_i with the variable x_i.

Definition 1. The *edge ideal* $I(G)$ associated to the graph G is the ideal of R generated by the set of square-free monomials $x_i x_j$ such that v_i is adjacent to v_j, that is,

$$I(G) = (\{x_i x_j \mid \{v_i, v_j\} \in E(G)\}).$$

If all the vertices of G are isolated we set $I(G) = (0)$.

Note that the non zero edge ideals are precisely the ideals of R generated by square-free monomials of degree two.

Definition 2. A local ring (R, m) is called *Cohen-Macaulay* if depth(R) = dim(R). If R is non local and R_p is a *Cohen-Macaulay* local ring for all $p \in Spec(R)$, then we say that R is a *Cohen-Macaulay* ring.

Definition 3. The graph G is said to be *Cohen-Macaulay* over the field k if $R / I(G)$ is a Cohen-Macaulay ring

Definition 4. Let G be a graph with vertex set V. A subset $A \subset V$ is a *minimal vertex cover* for G if:
1. every edge of G is incident with one vertex in A, and
2. there is no proper subset of A with the first property.

One of the purposes here is to show how large classes of Cohen-Macaulay graphs can be produced and to show some obstruction for a graph to be Cohen-Macaulay.

The First Construction

Let H be a graph with vertex set $V(H) = \{x_1, ..., x_n, z, w\}$ and J its edge ideal. Assume that z is adjacent to w with $\deg(z) \geq 2$ and $\deg(w)=1$. We label the vertices of H such that $x_1, ..., x_k, w$ are the vertices of H adjacent of z, as shown in the figure 1.

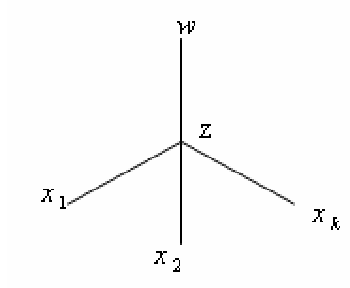

Fig. 1. The first construction

The next two results describe how the Cohen-Macaulay property of H relates to that of the two subgraphs $G = H \setminus \{z, w\}$ and $F = G \setminus \{x_1, ..., x_k\}$. One has the equalities:

$$J = (I, x_1 z, ..., x_k z, zw) \text{ and } (I, x_1, ..., x_k) = (L, x_1, ..., x_k),$$

where $I = I(G)$ and $L = I(F)$ are the edge ideals of G and F respectively.

Assume H is unmixed with height of J equal to $g + 1$. Since z is not isolated, there is a minimal prime p over I containing $\{x_1, ..., x_k\}$ and such that

$$\mathrm{ht}(I) = \mathrm{ht}(p) = g.$$

It is not difficult to prove that $k < n$ and $\deg(x_i) \geq 2$ for $1 \leq i \leq k$.

Proposition 1. If H is a *Cohen-Macaulay* graph, then F and G are Cohen-Macaulay graphs.

Proof. Let $A = k[x_1, ..., x_n]$ and $R = A[z, w]$. There exists a homogeneous system of parameters $\{f_1, ..., f_d\}$ for A/I, where $f_i \in A_+$ for all i. Because of the hypothesis and the equalities

$$z(z - w) + zw = z^2 \text{ and } w(w - z) + zw = w^2,$$

the set $\{f_1, ..., f_d, z - w\}$ is a regular system of parameters for R/J.

Hence $\{f_1, ..., f_d\}$ is a regular sequence on R/I, that is, G is Cohen-Macaulay.

Now we consider the sequence

$$0 \longrightarrow \frac{R}{(I, x_1, ..., x_k, w)}(-1) \xrightarrow{z} \frac{R}{J} \xrightarrow{\psi} \frac{R}{(I, z)} \longrightarrow 0,$$

where the first map is the multiplication by z and ψ is induced by a projection.

By the depth lemma one has $n - g + 1 \leq \operatorname{depth} \dfrac{R}{(I, x_1, ..., x_k, w)}$, where $g = \mathrm{ht}(J)$.

Since $(I, x_1, ..., x_k, w) = (L, x_1, ..., x_k, w)$, F is Cohen-Macaulay graph.

Proposition 2. If F and G are Cohen-Macaulay graphs and $\{x_1, ..., x_k\}$ form a part of a minimal vertex cover for G, then H is Cohen-Macaulay graph.

Proof. Consider the exact sequence

$$0 \longrightarrow \frac{R}{(I, x_1, ...x_k, w)}(-1) \xrightarrow{z} \frac{R}{J} \xrightarrow{\psi} \frac{R}{(I, z)} \longrightarrow 0$$

Since $\frac{R}{(I, x_1, ..., x_k, w)}$ and $\frac{R}{(I, z)}$ are Cohen-Macaulay rings, then $\frac{R}{J}$ is Cohen-Macaulay ring, that is, H is Cohen-Macaulay graph.

Corollary 3. If G is Cohen-Macaulay graph and $\{x_1, ..., x_k\}$ is the minimal vertex cover for G, then H is Cohen-Macaulay graph.

Proof. Since $I(F) = (0)$ result F is Cohen-Macaulay graph and we apply proposition 2.

The Second Construction

For the discussion of the second construction we change our notation. Let H be a graph on the vertex set $V(H) = \{x_1, ..., x_n, z\}$ so that $\{x_1, ..., x_k\}$ be the vertex of H adjacent to z.

We may assume $\deg(x_i) \geq 2$ for $1 \leq i \leq k$ and $\deg(z) \geq 2$.

Setting $G = H \setminus \{z\}$ and $F = G \setminus \{x_1, ..., x_k\}$, notice that the ideals $J = I(H)$, $I = I(G)$ and $L = I(F)$ associated to H, G and F respectively are related by the equalities :

$$J = (I, x_1 z, \dots, x_k z) \text{ and } (I, x_1, ..., x_k) = (L, x_1, ..., x_k).$$

Proposition 4. If H is Cohen-Macaulay graph, then F is Cohen-Macaulay graph.

Proof. Let $A = k[x_1, ..., x_n]$ and $R = A[z]$ and ht $J = g + 1$. The polynomial

$f = z - x_1 - ... - x_k$ is regular on R/J because it is not contained in any associated prime of J. There is a sequence $\{f_1, \dots, f_{n-g-1}\}$ regular on $\frac{R}{J}$ so that $\{f_1, f_2, ..., f_{n-g-1}\} \subset A_+$. Observe that $\{f_1, ..., f_{n-g-1}\}$ is in fact a regular sequence on $\frac{A}{I}$, which gives depth $(\frac{A}{I}) \geq n - g - 1$. Now, we use the sequence

$$0 \to \frac{R}{(I, x_1, ..., x_k)}(-1) \xrightarrow{z} \frac{R}{J} \to \frac{R}{(I, z)} \to 0$$

and ht $(I, x_1, ..., x_k) = g + 1$ to conclude that F is Cohen-Macaulay graph.

Proposition 5. Assume $x_1, \dots, x_k$ do not form a part of a minimal vertex cover for G and ht $(I, x_1, ..., x_k) = \text{ht}(I) + 1$. If F and G are Cohen-Macaulay graphs, then H is Cohen-Macaulay graph.

Proof. The assumption on $\{x_1, \dots, x_k\}$ forces ht $(J) = \text{ht}(I) + 1$.

From the exact sequence:

$$0 \longrightarrow \frac{R}{(I, x_1, \ldots, x_k)}(-1) \xrightarrow{z} \frac{R}{J} \longrightarrow \frac{R}{(I, z)} \longrightarrow 0$$

we obtain that H is Cohen-Macaulay graph.

Corollary 6. If G is Cohen-Macaulay graph and $\{x_1, \ldots, x_{k-1}\}$ is a minimal vertex cover for G, then H is Cohen-Macaulay graph.

A good property of Cohen-Macaulay graphs is its additivity with respect to connected components.

Lemma 7. Let $R_1 = k[x_1, \ldots, x_n]$ and $R_2 = k[y_1, \ldots, y_m]$ be two polynomial rings over a field k and $R = k[x_1, \ldots, x_n, y_1, \ldots, y_m]$. If I_1 and I_2 are graded ideals in R_1 and R_2 respectively, then

$$\text{depth}\left(\frac{R_1}{I_1}\right) + \text{depth}\left(\frac{R_2}{I_2}\right) = \text{depth}\left(\frac{R}{I_1 + I_2}\right).$$

Proof. Because $\frac{R_2}{I_2} \otimes_k R_1 = \frac{R}{I_2 R}$ is a flat R_1 module the equality holds from a general property of tensor products.

Proposition 8. If G is a graph and $G_1, \ldots, G_n$ its connected components, then G is Cohen-Macaulay graph if and only if $G_1, \ldots, G_n$ are Cohen-Macaulay graphs.

Proof. Evident from Lemma 7.

References

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Grafuri Cohen-Macaulay

Rezumat

Scopul acestui articol este de a prezenta construcţia unor clase de grafuri Cohen-Macaulay şi câteva exemple. De asemenea, este prezentată o proprietate foarte importantă a inelelor Cohen-Macaulay şi anume ca fibra să fie Cohen-Macaulay.